On coformality of moment-angle complexes

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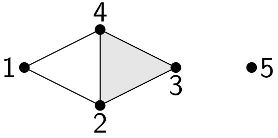
Plan

- Moment-angle complexes
- (Co)formality
- Known results
- Non-coformal moment-angle complexes
- Coformal moment-angle complexes

Simplicial complexes

A simplicial complex $\mathcal{K}$ on vertex set V is a family of subsets $I \subset V$ called simplices, that satisfies two conditions:

- $\{v\} \in \mathcal{K}$ for every $v \in V$ ("no ghost vertices");
- if $J \in \mathcal{K}$ and $I \subset J$, then $I \in \mathcal{K}$.



In the picture: $V = \{1,2,3,4,5\}$, $\mathcal{K} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}\}$.

Flag complexes and flagification

A complex $\mathcal{K}$ is flag if any set of pairwise adjacent vertices is a simplex. For any $\mathcal{K}$, there is unique flag complex $\mathcal{K}^f$ with the same 1-skeleton, called the flagification of $\mathcal{K}$. More explicitly:

$$\mathcal{K}^f := \{J \subset [m] : \ \{i,j\} \in \mathcal{K}, \ \forall i,j \in J\} \supset \mathcal{K}.$$

Polyhedral products

Definition

(X, A) a pair of spaces, $\mathcal{K}$ a simplicial complex on $[m] := \{1, \dots, m\}$ $\Rightarrow$ define

$$(X,A)^{\mathcal{K}} := \bigcup_{J \in \mathcal{K}} \left(\prod_{i \in J} X \times \prod_{i \in [m] \setminus J} A \right) \subset X^m.$$

A simple example: $\mathcal{K} = \bullet\bullet \;\Rightarrow\; (X,A)^{\mathcal{K}} = (X \times A) \cup (A \times X) \subset X \times X$.

Important special cases

- Davis–Januszkiewicz spaces $\mathrm{DJ}(\mathcal{K}) := (\mathbb{C}\mathrm{P}^\infty, \mathrm{pt})^{\mathcal{K}}$;
- Moment-angle complexes $\mathcal{Z}_{\mathcal{K}} := (D^2, S^1)^{\mathcal{K}}$.

Theorem (Buchstaber–Panov, 2000)

There is a homotopy fibration $\mathcal{Z}_{\mathcal{K}} \to \mathrm{DJ}(\mathcal{K}) \to (\mathbb{C}\mathrm{P}^\infty)^m$.

Quasi-isomorphisms

By a dg-algebra (A, d) we mean an associative differential graded algebra $(d^2 = 0,\ a \cdot (b \cdot c) = (a \cdot b) \cdot c,\ d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b))$.

Definition

dg-algebras (A, d_A) and (B, d_B) are quasi-isomorphic if there are dg-algebras (Γ_i, d_i) and maps of dg-algebras

$$(A, d_A) \xleftarrow{f_1} (\Gamma_1, d_1) \xrightarrow{f_2} (\Gamma_2, d_2) \xleftarrow{f_3} \dots \xrightarrow{f_n} (B, d_B)\ ,$$

such that the induced maps in homology

$$H(A, d_A) \xleftarrow{(f_1)_*} H(\Gamma_1, d_1) \xrightarrow{(f_2)_*} H(\Gamma_2, d_2) \xleftarrow{(f_3)_*} \dots \xrightarrow{(f_n)_*} H(B, d_B)$$

are isomorphisms.

The condition $H(A, d_A) \simeq H(B, d_B)$ is necessary. But not sufficient!

Formality and coformality

Let $\mathbb{k}$ be a commutative ring with unit. For a topological space X, there are two dg-algebras over $\mathbb{k}$: $C^*(X;\mathbb{k})$ with the cup product and $C_*(\Omega X;\mathbb{k})$ with the Pontryagin product.

Definition

- X is $\mathbb{k}$-formal if $C^*(X;\mathbb{k})$ is quasi-isomorphic to $H^*(X;\mathbb{k})$.
- X is $\mathbb{k}$-coformal if $C_*(\Omega X;\mathbb{k})$ is quasi-isomorphic to $H_*(\Omega X;\mathbb{k})$.

(Coformality) in rational homotopy theory

For $\mathbb{k}=\mathbb{Q}$, the classic definitions of formality and coformality are different (in terms of Sullivan models and Quillen models). Our approach is equivalent to the classic one: see [Saleh'17, arXiv:1609.02540].

What is known about (co)formality of $\mathcal{Z}_\mathcal{K}$ and $\mathrm{DJ}(\mathcal{K})$?

- (Notbohm,Ray'03) $\mathrm{DJ}(\mathcal{K})$ is $\mathbb{k}$-formal for any $\mathbb{k}$ and $\mathcal{K}$.
  - Proof: $\mathbb{k}$-formality of $\mathbb{C}\mathrm{P}^\infty$ + some colimit arguments.
- (Panov,Ray'07) $\mathrm{DJ}(\mathcal{K})$ is $\mathbb{Q}$-coformal $\Leftrightarrow$ $\mathcal{K}$ is flag.
  - Obstructions to coformality: higher Samelson products in $\pi_*(\Omega\mathrm{DJ}(\mathcal{K}))$.
- (Baskakov'03; Denham,Suciu'07; Buchstaber,Limonchenko'19; Zhuravleva'19; ...) Several families of simplicial complexes, such that $\mathcal{Z}_\mathcal{K}$ is not formal.
  - Obstructions to formality: higher Massey products in $H^*(\mathcal{Z}_\mathcal{K};\mathbb{k})$.
- (Bosio,Meersseman'04; Grbić,Theriault'07; Gitler, Lopez de Medrano'12; Iriye,Kishimoto'14; Fan,Chen,Ma,Wang'14; ...) Several families of simplicial complexes, such that $\mathcal{Z}_\mathcal{K}$ is formal and coformal over any $\mathbb{k}$.
  - Proof: the homotopy types of such $\mathcal{Z}_\mathcal{K}$ are known (wedges of spheres, connected sums of sphere products)
- No known examples of non-coformal moment-angle complexes!

Main result

Notation: $\mathcal{K}_J$ is the full subcomplex of $\mathcal{K}$ on vertex set $J \subset [m]$.

Theorem (V.'22)

If $\mathcal{Z}_\mathcal{K}$ is coformal over a field $\Bbbk$, then the map $H_*(\mathcal{K}_J;\Bbbk) \to H_*(\mathcal{K}^f_J;\Bbbk)$ is surjective for every $J \subset [m]$.

Therefore: if we obtain "new classes in homology" when passing from $\mathcal{K}$ to $\mathcal{K}^f$, then $\mathcal{Z}_\mathcal{K}$ is not coformal.

Example: 1-skeleton of a cross-polytope

Let $\mathcal{K} = \mathrm{sk}_1(S^0 * S^0 * S^0)$, so $\mathcal{K}^f = S^0 * S^0 * S^0$. Hence $H_2(\mathcal{K};\Bbbk) = 0$, $H_2(\mathcal{K}^f;\Bbbk) = \Bbbk$, thus $\mathcal{Z}_\mathcal{K}$ is not coformal.
Another proof: the homotopy equivalence

$$\mathcal{Z}_\mathcal{K} \simeq (S^5)^{\vee 8} \vee (S^6)^{\vee 24} \vee (S^7)^{\vee 24} \vee (S^8)^{\vee 7} \vee \mathrm{FW}(S^3, S^3, S^3),$$

where FW is the fat wedge. Fat wedges are not coformal.

Sketch of the proof

For any 1-connected space X and any field $\Bbbk$ we have the Milnor–Moore spectral sequence

$$E^2_{p,q} \cong \mathrm{Tor}_p^{H_*(\Omega X;\Bbbk)}(\Bbbk,\Bbbk)_q, \quad E^\infty \simeq H_*(X;\Bbbk).$$

Fact: if X is $\Bbbk$-coformal, then $E^2 = E^\infty$.

For $X = \mathcal{Z}_\mathcal{K}$ there is an additional grading $E^r_{p,q} = \bigoplus_{q=-n+2|\alpha|} E^r_{p,-n,2\alpha}$.

We compute some graded components:

$$E^2_{p,-|J|,2J} \cong \widetilde{H}_{p-1}(\mathcal{K}^f_J;\Bbbk); \quad E^\infty_{p,-|J|,2J} \cong \mathrm{Im}\left(\widetilde{H}_{p-1}(\mathcal{K}_J;\Bbbk) \to \widetilde{H}_{p-1}(\mathcal{K}^f_J;\Bbbk)\right).$$

Therefore: if $E^2 = E^\infty$, then $\widetilde{H}_{p-1}(\mathcal{K}_J;\Bbbk) \twoheadrightarrow \widetilde{H}_{p-1}(\mathcal{K}^f_J;\Bbbk)$. □

An open question

$E^2 \neq E^\infty$ implies that there are non-trivial differentials. Can we describe them using higher commutator products in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ and/or higher Whitehead products in $\pi_*(\mathcal{Z}_\mathcal{K})$?

For the 1-skeleton of the cross-polytope: yes, the only non-trivial differential in MMSS corresponds to the higher product $[[u_1,u_2],[u_3,u_4],[u_5,u_6]]$.

Coformality of $\mathcal{Z}_\mathcal{K}$, flag case

The following is proved using rational homotopy theory (Sullivan models):

Proposition (R.Huang'21)

Let $F \to E \to B$ be a fibration of simply connected spaces, such that

1. $\pi_*(F)\otimes\mathbb{Q} \to \pi_*(E)\otimes\mathbb{Q}$ is injective ("TNHZ condition");
2. E is $\mathbb{Q}$-coformal.

Then F is $\mathbb{Q}$-coformal.

We apply this result:

- $\mathcal{Z}_\mathcal{K} \to \mathrm{DJ}(\mathcal{K}) \to (\mathbb{C}\mathrm{P}^\infty)^m$ is a fibration (Buchstaber, Panov'00);
- $\pi_*(\mathcal{Z}_\mathcal{K}) \to \pi_*(\mathrm{DJ}(\mathcal{K}))$ is injective, since the fibration $\Omega\mathcal{Z}_\mathcal{K} \to \Omega\mathrm{DJ}(\mathcal{K}) \to \Omega(\mathbb{C}\mathrm{P}^\infty)^m$ splits;
- $\mathrm{DJ}(\mathcal{K})$ is $\mathbb{Q}$-coformal for flag $\mathcal{K}$ (Panov,Ray'07).

Therefore, moment-angle complexes for flag $\mathcal{K}$ are $\mathbb{Q}$-coformal.
Their $\mathbb{Z}_p$-coformality is an open question.

References

- V. M. Buchstaber and T. E. Panov. *Toric topology*, volume 204 of *Mathematical Surveys and Monographs*.
  AMS, Providence, RI (2015). arXiv:1210.2368

- S. A. Abramyan, T.E. Panov. Higher Whitehead products for moment-angle complexes and substitutions of simplicial complexes.
  Proc. Steklov. Math, 305(2019), 7–28. arXiv:1901.07918

- R. Huang. Coformality around fibrations and cofibrations.
  Preprint (2021). arXiv:2108.08446

- F. E. Vylegzhanin. Pontryagin algebras and the LS-category of moment-angle complexes in the flag case.
  Proc. Steklov. Math., 317(2022), to appear. arXiv:2203.08791

- F. Vylegzhanin. Milnor-Moore spectral sequence for moment-angle complexes.
  In preparation... (2023?)