Discrete complex analysis

Convergence results

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Discretizations of complex analysis

Discrete complex analysis

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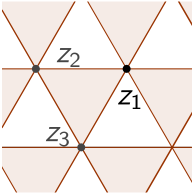
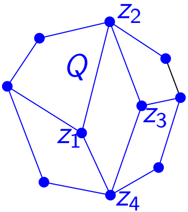
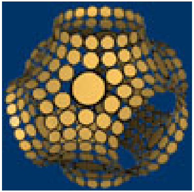






. . .

$f(z_1)+f(z_2)+f(z_3)=0$

$$\frac{f(z_1)-f(z_3)}{z_1-z_3}=\frac{f(z_2)-f(z_4)}{z_2-z_4}$$

. . .

Dynnikov–Novikov ↓ integrable systems

Isaacs, Ferrand, . . . ↓ numerical analysis, network theory, statistical physics

Thurston ↓ conformal geometry

Overview

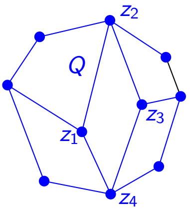
1. Discrete analytic functions in a planar domain
2. Discrete analytic functions in a Riemann surface
3. Convergence via energy estimates

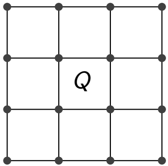
1
Discrete analytic functions in a planar domain

A graph $Q \subset \mathbb{C}$ is a *quadrilateral lattice* $\Leftrightarrow$ each bounded face is a quadrilateral
A function $f\colon Q \to \mathbb{C}$ is *discrete analytic* $\Leftrightarrow$

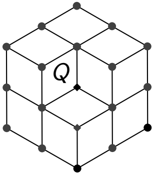
$$\frac{f(z_1)-f(z_3)}{z_1-z_3} = \frac{f(z_2)-f(z_4)}{z_2-z_4}$$

for each face $z_1 z_2 z_3 z_4$ with the vertices listed clockwise. $\mathrm{Re}\, f$ is called *discrete harmonic.*

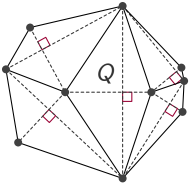




square lattice
Isaacs,Ferrand (1940s)



rhombic lattice
Duffin (1960s)

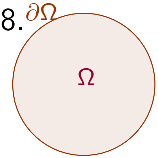


orthogonal lattice
Mercat (2000s)

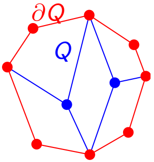
The Dirichlet boundary value problem

Problem. Prove convergence of discrete harmonic functions to their continuous counterparts as $h \to 0$.

- *Square lattices*, C^0: Lusternik, 1926.
- *Square lattices*, C^∞: Courant–Friedrichs–Lewy, 1928.
- *Rhombic lattices*, C^0: Ciarlet–Raviart, 1973 (implicitly).
- *Rhombic lattices*, C^1: Chelkak–Smirnov, 2008.



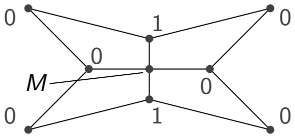
The *Dirichlet problem* in a domain Ω is to find a continuous function $u_{\Omega,g}\colon \mathrm{Cl}\Omega \to \mathbb{R}$ having given boundary values $g\colon \partial\Omega \to \mathbb{R}$ and such that $\Delta u_{\Omega,g} = 0$ in Ω.



The *Dirichlet problem* on Q is to find a discrete harmonic function $u_{Q,g}\colon Q \to \mathbb{R}$ having given boundary values $g\colon \partial Q \to \mathbb{R}$.

Existence and Uniqueness Theorem (S. 2011). *The Dirichlet problem on any finite quadrilateral lattice has a unique solution.*

Example (Tikhomirov, 2011): no maximum principle!



z	0	$\pm i$	$\pm\cot\frac{\pi}{8}$	$\pm\sqrt{2}M(\cot\frac{\pi}{8}+i)$	$\pm\sqrt{2}M(\cot\frac{\pi}{8}-i)$
$f(z)$	$M(1+i)$	1	0	0	$2Mi$
$\mathrm{Re} f(z)$	M	1	0	0	0

Both $f(z)$ and the shape of Q depends on a prameter M.

Convergence Theorem for the Dirichlet Problem

A sequence $\{Q_n\}$ is *nondegenerate uniform* $\Leftrightarrow \exists\, \mathrm{const} > 0$:

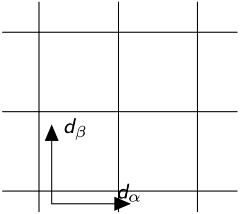
- the angle between the diagonals and the ratio of the diagonals in each quadrilateral face are $> \mathrm{const}$,
- the number of vertices in each disk of radius $\mathrm{Size}(Q_n)$ is $< \mathrm{const}^{-1}$, where $\mathrm{Size}(Q_n)$:=maximal edge length.

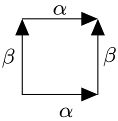
Convergence Theorem for BVP (S. 2013). *Let $\Omega \subset \mathbb{C}$ be a bounded simply-connected domain. Let $g : \mathbb{C} \to \mathbb{R}$ be a smooth function. Take a nondegenerate uniform sequence of finite orthogonal lattices $\{Q_n\}$ such that $\mathrm{Size}(Q_n)$, $\mathrm{Dist}(\partial Q_n, \partial\Omega) \to 0$.* ***Then the solution $u_{Q_n,g} : Q_n \to \mathbb{R}$ of the Dirichlet problem on Q_n uniformly converges to the solution $u_{\Omega,g} : \Omega \to \mathbb{R}$ of the Dirichlet problem in Ω.***

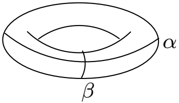
2
Discrete analytic functions in Riemann surfaces

Riemann surfaces

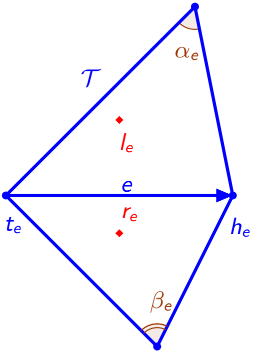
Riemann surface	Analytic functions
planar domain	functions $u(x,y)+iv(x,y)$ s.t. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$
quotient $\mathbb{C}$ by a lattice	doubly periodic analytic functions
complex algebraic curve $a_{nm}z^n w^m+\dots+a_{00}=0$	analytic functions in both w and z
polyhedral surface	continuous functions which are analytic on each face







$\mathcal{R}$	a polyhedral surface
$\mathcal{T}$	its triangulation
$\mathcal{T}^0$	the set of vertices
$\vec{\mathcal{T}}^1$	the set of oriented edges
$\mathcal{T}^2$	the set faces



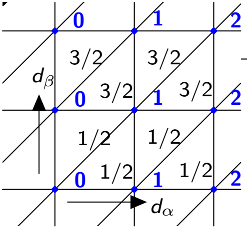
A *discrete analytic function* is a pair $(u\colon \mathcal{T}^0 \to \mathbb{R}, v\colon \mathcal{T}^2 \to \mathbb{R})$ such that $\forall e \in \vec{\mathcal{T}}^1$

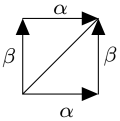
$$v(l_e) - v(r_e) = \frac{\cot \alpha_e + \cot \beta_e}{2}(u(h_e) - u(t_e)).$$

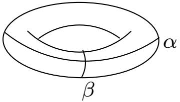
(Duffin, Pinkall–Polthier, Desbrun–Meyer–Schröder, Mercat)

Remark. $\mathcal{T}$ is a *Delauney* triangulation of $\mathbb{R}^2 \Rightarrow u \sqcup iv$ is discrete analytic on Q (in the sense of **Part 1** of the slides).

$p\colon \widetilde{\mathcal{R}} \to \mathcal{R}$	the universal covering
$\{\alpha, \beta\}$	the basis of $\pi_1(\mathcal{R})$
$\{d_\alpha, d_\beta\}$	the automorphisms of p







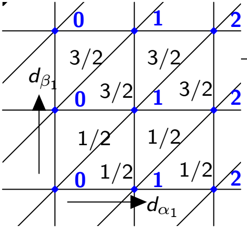
$$\widetilde{\mathcal{T}} \quad \xrightarrow{p} \quad \mathcal{T} \quad \approx \quad S^1 \times S^1$$

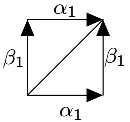
A *discrete Abelian integral of the 1st kind* with periods $A, B \in \mathbb{C}$ is a discrete analytic function $(\mathrm{Re} f : \widetilde{\mathcal{T}}^0 \to \mathbb{R}, \mathrm{Im} f : \widetilde{\mathcal{T}}^2 \to \mathbb{R})$ such that $\forall z \in \widetilde{\mathcal{T}}^0, \forall w \in \widetilde{\mathcal{T}}^2$

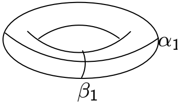
$$[\mathrm{Re} f](d_\alpha z) - [\mathrm{Re} f](z) = \mathrm{Re}\, A; \quad [\mathrm{Re} f](d_\beta z) - [\mathrm{Re} f](z) = \mathrm{Re}\, B;$$
$$[\mathrm{Im} f](d_\alpha w) - [\mathrm{Im} f](w) = \mathrm{Im}\, A; \quad [\mathrm{Im} f](d_\beta w) - [\mathrm{Im} f](w) = \mathrm{Im}\, B.$$

Discrete Abelian integrals of the 1st kind

$p\colon \widetilde{\mathcal{R}} \to \mathcal{R}$	the universal covering
$\{\alpha_k, \beta_k\}_{k=1}^{g}$	the basis of $\pi_1(\mathcal{R})$
$\{d_{\alpha_k}, d_{\beta_k}\}_{k=1}^{g}$	the automorphisms of p







$$\widetilde{\mathcal{T}} \quad \xrightarrow{p} \quad \mathcal{T} \quad \approx \quad S^1 \times S^1$$

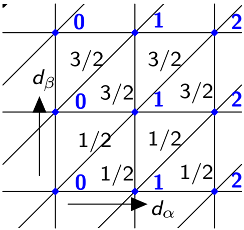
A *discrete Abelian integral of the 1st kind* with periods $A_1, \dots, A_g, B_1, \dots, B_g \in \mathbb{C}$ is a discrete analytic function $(\mathrm{Re} f : \widetilde{\mathcal{T}}^0 \to \mathbb{R}, \mathrm{Im} f : \widetilde{\mathcal{T}}^2 \to \mathbb{R})$ such that $\forall z \in \widetilde{\mathcal{T}}^0, \forall w \in \widetilde{\mathcal{T}}^2$

$$\mathrm{Re} f(d_{\alpha_k} z) - \mathrm{Re} f(z) = \mathrm{Re}\, A_k; \quad \mathrm{Re} f(d_{\beta_k} z) - \mathrm{Re} f(z) = \mathrm{Re}\, B_k;$$
$$\mathrm{Im} f(d_{\alpha_k} w) - \mathrm{Im} f(w) = \mathrm{Im}\, A_k; \quad \mathrm{Im} f(d_{\beta_k} w) - \mathrm{Im} f(w) = \mathrm{Im}\, B_k.$$

Period matrix

Existence & Uniqueness Theorem (Bobenko–S. 2012)
$\forall A \in \mathbb{C}$ *there is a discrete Abelian integral of the 1st kind with the A-period A. It is unique up to constant.*

The *discrete period matrix* $\Pi_{\mathcal{T}}$ (*period matrix* $\Pi_{\mathcal{R}}$) is the B-period of the discrete Abelian integral (Abelian integral) of the 1st kind with the A-period 1.
It is a 1×1 matrix for a surface of genus 1.



$$\Pi_{\mathcal{T}} = i = \Pi_{\mathcal{R}}$$

Notation.

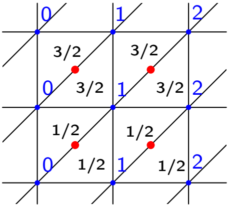
$\gamma_z := 2\pi(\text{the sum of angles meeting at } z)^{-1}$

$\gamma_z > 1 \Leftrightarrow$ "curvature" > 0

$\gamma_{\mathcal{R}} := \min_{z \in \mathcal{T}^0}\{1, \gamma_z\}$

Existence & Uniqueness Theorem (Bobenko–S. 2012)
For any numbers $A_1, \dots, A_g \in \mathbb{C}$ there exist a discrete Abelian integral of the 1st kind with A-periods $A_1, \dots, A_g$. It is unique up to constant.

Let $\phi^l_{\mathcal{T}} = (\mathrm{Re}\,\phi^l_{\mathcal{T}} \colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}, \mathrm{Im}\,\phi^l_{\mathcal{T}} \colon \widetilde{\mathcal{T}}^2 \to \mathbb{R})$ be the unique (up to constant) discrete Abelian integral of the 1st kind with A-periods $A_k = \delta_{kl}$.
The *discrete period matrix* $\Pi_{\mathcal{T}}$ is the $g \times g$ matrix whose columns are the B-periods of $\phi^1_{\mathcal{T}}, \dots, \phi^g_{\mathcal{T}}$.



Example. For $\mathcal{R} = \mathbb{C}/(\mathbb{Z} + \eta\mathbb{Z})$:

$\mathrm{Re}\,\phi^1_{\mathcal{T}}(z) = \mathrm{Re}\,z$,
$\mathrm{Im}\,\phi^1_{\mathcal{T}}(w) = \mathrm{Im}\,w^*$,
where w^* is the circumcenter of a face w.

The complex structure on polyhedral surfaces

Polyhedral metric $\rightsquigarrow$ complex structure

Identify each face $w \in \widetilde{T}^2$ with a triangle in $\mathbb{C}$ by an orientation-preserving isometry.

A function $f\colon \widetilde{\mathcal{R}} \to \mathbb{C}$ is *analytic*, if it is continuous and its restriction to the interior of each face is analytic.

Let $\phi^l_{\mathcal{R}}\colon \widetilde{\mathcal{R}} \to \mathbb{C}$ be the unique (up to constant) Abelian integral of the 1st kind with A-periods $A_k = \delta_{kl}$.

The *period matrix* $\Pi_{\mathcal{R}}$ is the $g \times g$ matrix whose columns are the B-periods of $\phi^1_{\mathcal{R}}, \dots, \phi^g_{\mathcal{R}}$.

$\gamma_z := 2\pi(\text{the sum of angles meeting at } z)^{-1}$

$\gamma_z > 1 \Leftrightarrow$ "curvature" > 0

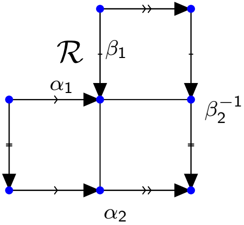
$\gamma_{\mathcal{R}} := \min_{z \in T^0}\{1, \gamma_z\}$

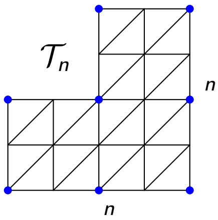
Convergence Theorem for Period Matrices (Bobenko–S. 2013) $\forall\delta > 0\ \exists \mathrm{Const}_{\delta,\mathcal{R}}, \mathrm{const}_{\delta,\mathcal{R}} > 0$ *such that for any triangulation $\mathcal{T}$ of $\mathcal{R}$ with the maximal edge length $h < \mathrm{const}_{\delta,\mathcal{R}}$ and with the minimal face angle $> \delta$ we have*

$$\|\Pi_{\mathcal{T}} - \Pi_{\mathcal{R}}\| \le \mathrm{Const}_{\delta,\mathcal{R}} \cdot \begin{cases} h, & \text{if } \gamma_{\mathcal{R}} > 1/2; \\ h|\log h|, & \text{if } \gamma_{\mathcal{R}} = 1/2; \\ h^{2\gamma_{\mathcal{R}}}, & \text{if } \gamma_{\mathcal{R}} < 1/2. \end{cases}$$

Corollary. ***The discrete period matrices*** *of a sequence of triangulations of the surface with the maximal edge length tending to zero and with face angles bounded from zero* ***converge to the period matrix*** *of the surface.*

Model surface:





Computations using a software by S. Tikhomirov:

n	$\|\Pi_{\mathcal{T}_n} - \Pi_{\mathcal{R}}\|$	$\|\Pi_{\mathcal{T}_n} - \Pi_{\mathcal{R}}\| \cdot h^{-2\gamma_{\mathcal{R}}}$
8	0.611	1.22
16	0.363	1.15
32	0.220	1.11
64	0.136	1.08
128	0.084	1.07
256	0.053	1.06

A sequence $\{\mathcal{T}_n\}$ is *nondegenerate uniform* $\Leftrightarrow \exists \mathrm{const} > 0$:

- the minimal face angle is $> \mathrm{const}$;
- $\forall e \in \vec{\mathcal{T}}_n^1$ we have $\alpha_e + \beta_e < \pi - \mathrm{const}$;
- the number of vertices in an arbitrary disk of radius equal to the maximal edge length ($=: \mathrm{Size}(\mathcal{T}_n)$) is $< \mathrm{const}^{-1}$.

Convergence Theorem for Abelian integrals (Bobenko–S. 2013) *Let $\{\mathcal{T}_n\}$ be a nondegenerate uniform sequence of triangulations of $\mathcal{R}$ with $\mathrm{Size}(\mathcal{T}_n) \to 0$. Let $z_n \in \widetilde{\mathcal{T}}_n^0$ converge to $z_0 \in \widetilde{\mathcal{R}}$ and $w_n \in \widetilde{\mathcal{T}}_n^2$ contain z_n.* ***Then the discrete Abelian integrals of the 1st kind*** $\phi^l_{\mathcal{T}_n} = (\mathrm{Re}\,\phi^l_{\mathcal{T}_n} : \widetilde{\mathcal{T}}_n^0 \to \mathbb{R}, \mathrm{Im}\,\phi^l_{\mathcal{T}_n} : \widetilde{\mathcal{T}}_n^2 \to \mathbb{R})$ ***normalized by*** $\mathrm{Re}\,\phi^l_{\mathcal{T}}(z_n) = \mathrm{Im}\,\phi^l_{\mathcal{T}}(w_n) = 0$ ***converge to the Abelian integral of the 1st kind*** $\phi^l_{\mathcal{R}} : \widetilde{\mathcal{R}} \to \mathbb{C}$ ***normalized by*** $\phi^l_{\mathcal{R}}(z_0) = 0$ *uniformly on compact subsets.*

Discrete Riemann–Roch theorem

A *discrete meromorphic function* is an arbitrary pair $(\mathrm{Re} f : \mathcal{T}^0 \to \mathbb{R}, \mathrm{Im} f : \mathcal{T}^2 \to \mathbb{R})$.

$\mathrm{res}_e f := \mathrm{Im} f(r_e) - \mathrm{Im} f(l_e) + \nu(e)\mathrm{Re} f(h_e) - \nu(e)\mathrm{Re} f(t_e)$

A *divisor* is a map $D \colon \mathcal{T}^0 \sqcup \mathcal{T}^1 \sqcup \mathcal{T}^2 \to \{0, \pm 1\}$.

$(f) := I_{\mathrm{Re} f=0} - I_{\mathrm{res}_e f \neq 0} + I_{\mathrm{Im} f=0}$; $\quad l(D) := \dim\{f : (f) \geq D\}$

A *discrete Abelian differential* is an odd map $\omega \colon \vec{\mathcal{T}}^1 \to \mathbb{R}$.

$\mathrm{res}_w \, \omega := \sum_{e \in \vec{\mathcal{T}}^1 : l_e = w} \omega(e)$; $\quad \mathrm{res}_z \, \omega := i \sum_{e \in \vec{\mathcal{T}}^1 : h_e = z} \nu(e)\omega(e)$.

$(\omega) := -I_{\mathrm{res}_z \omega \neq 0} + I_{\omega=0} - I_{\mathrm{res}_w \omega \neq 0}$; $i(D) := \dim\{\omega : (\omega) \geq D\}$

D is *admissible* $\Leftrightarrow (-1)^k D(\mathcal{T}^k) \leq 0$; $\quad \deg D := \sum_z D(z)$.

Discrete Riemann–Roch Theorem (Bobenko–S. 2012)

For admissible divisors D on a triangulated surface of genus g

$$l(-D) = \deg D - 2g + 2 + i(D).$$

3
Convergence via energy estimates

Main concept: energy

The *energy* of a function $u\colon \Omega \to \mathbb{R}$ is $E_\Omega(u) := \int_\Omega |\nabla u|^2 dA$.
The *gradient* of a function $u\colon Q^0 \to \mathbb{R}$ at a face $z_1z_2z_3z_4$ is the unique vector $\nabla_Q u(z_1z_2z_3z_4) \in \mathbb{R}^2$ such that

$$\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_1z_3} = u(z_1) - u(z_3),$$
$$\nabla_Q u(z_1z_2z_3z_4) \cdot \overrightarrow{z_2z_4} = u(z_2) - u(z_4).$$

The *energy* of the function $u\colon Q^0 \to \mathbb{R}$ is

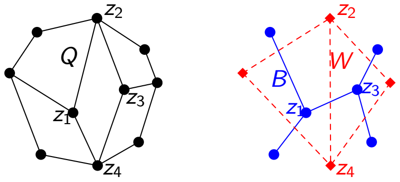
$$E_Q(u) := \sum_{z_1z_2z_3z_4 \subset Q} |\nabla_Q u(z_1z_2z_3z_4)|^2 \cdot \mathrm{Area}(z_1z_2z_3z_4).$$

Convexity Principle. *The energy $E_Q(u)$ is a strictly convex functional on the affine space $\mathbb{R}^{Q^0 - \partial Q}$ of functions $u\colon Q^0 \to \mathbb{R}$ having fixed values at the boundary ∂Q.*

Variational principle. *A function $u\colon Q^0 \to \mathbb{R}$ has minimal energy $E_Q(u)$ among all the functions with the same boundary values if and only if it is discrete harmonic.*

Physical interpretation

A *direct-current network*/*alternating-current network* is a connected graph with a marked subset of vertices (*boundary*) and a positive number/complex number with positive real part (*conductance*/*admittance*) assigned to each edge.



- The graph B is naturally an *alternating-current network*
- *Admittance* $c(z_1z_3) := i\frac{z_2-z_4}{z_1-z_3} \Rightarrow \mathrm{Re}\, c(z_1z_3) > 0$
- *Voltage* $V(z_1z_3) := f(z_1) - f(z_3)$
- *Current* $I(z_1z_3) := if(z_2) - if(z_4)$
- *Energy* $E(f) := \mathrm{Re} \sum_{z_1z_3} V(z_1z_3)\bar{I}(z_1z_3)$.

Energy Convergence Lemma. *Let $\partial\Omega$ be smooth and $\{Q_n\} \subset \Omega$ be a nondegenerate uniform sequence of quadrilateral lattices such that $\mathrm{Size}(Q_n), \mathrm{Dist}(\partial Q_n, \partial\Omega) \to 0$. Let $g\colon \mathbb{C} \to \mathbb{R}$ be a C^2 function. Then $E_{Q_n}(g\big|_{Q_n^0}) \to E_\Omega(g)$.*

Proof idea. *Discontinuous* piecewise-linear "interpolation": $I_Q g\colon z_1z_2z_3z_4 \to \mathbb{R}$ is the linear function s.t.

$$\begin{aligned} I_Q g(z_1) &= g(z_1),\\ I_Q g(z_3) &= g(z_3),\\ I_Q g(z_2) - I_Q g(z_4) &= g(z_2) - g(z_4). \end{aligned}$$

Thus $\nabla_Q g = \nabla I_Q g, E_Q(g) = E_{\Omega\cap Q}(I_Q g) \Rightarrow$ convergence.

Remark. Discontinuity $\Rightarrow$ usual finite element method helpless!

$u \colon B^0 \to \mathbb{R}$ is *Hölder* $\Leftrightarrow |u(z) - u(w)| \leq \mathrm{const} \cdot |z - w|^p$.

Discrete harmonic functions are Hölder:

- with $p = 1/2$ on *square* lattices (Courant et al 1928);
- with $p = 1$ on *rhombic* lattices (Chelkak–Smirnov, Kenyon 2008 Integrability!);
- with some p on *orthogonal* lattices (Saloff-Coste 1997).

Remark. (Informal meaning of integrability)
For any discrete analytic function $f \colon Q^0 \to \mathbb{C}$ its *primitive* $F(z_m) := \sum_{k=1}^{m-1} \frac{f(z_k)+f(z_{k+1})}{2}(z_{k+1} - z_k)$ is discrete analytic $\Leftrightarrow$ Q is *parallelogrammic*.

Problem (Chelkak, 2011). Are discrete harmonic functions Hölder with $p = 1$ on orthogonal lattices?

Equicontinuity Lemma. *Let Q be an orthogonal lattice. Let $u\colon Q^0 \to \mathbb{R}$ be a discrete harmonic function. Let $z, w \in B^0$ be two vertices with $|z-w| \geq \mathrm{Size}(Q)$. Let R be a square of side length $r > 3|z-w|$ with the center at $\frac{z+w}{2}$ and the sides parallel and orthogonal to zw. Then* $\exists \mathrm{Const}$: $|u(z)-u(w)| \leq$

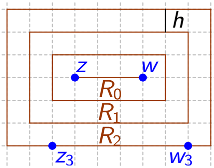
$$\mathrm{Const} \cdot E_Q(u)^{1/2} \cdot \log^{-1/2} \frac{r}{3|z-w|} + \max_{z',w' \in R \cap \partial Q \cap B^0} |u(z')-u(w')|.$$

Proof for a square lattice (cf. Lusternik 1926).

Assume $R \cap \partial Q = \emptyset$, $u(z) \geq u(w)$.

$R_m :=$ rectangle $2mh \times (2mh + |z-w|)$.

$m \leq \frac{r-|z-w|}{2h} \Rightarrow R_m \subset R \Rightarrow \exists z_m, w_m \in \partial R_m : u(z_m) \geq u(z), u(w_m) \leq u(w)$ Thus



$$E_Q(u) \geq \sum_{m=0}^{[(r-|z-w|)/2h]} \frac{|u(z_m)-u(w_m)|^2}{8m+2|z-w|/h} \geq \frac{|u(z)-u(w)|^2}{8} \log \frac{r}{3|z-w|}.$$

Approximation of laplacian

The *laplacian* of a function $u\colon Q^0 \to \mathbb{R}$: $[\Delta_Q u](z) := -\frac{\partial E_Q(u)}{\partial u(z)}$.

Remark. For a *parallelogrammic lattice* Q and a quadratic function g we have $\Delta_Q g = \Delta g$.

Laplacian Approximation Lemma *Let Q be a quadrilateral lattice, R be a square of side length $r > \mathrm{Size}(Q)$ inside ∂Q, and $g\colon \mathbb{C} \to \mathbb{R}$ be a smooth function. Then $\exists \mathrm{Const}$ such that*

$$\left| \sum_{z \in R \cap B^0} [\Delta_Q(g\,|_{Q^0})]\,(z) - \int_R \Delta g\, dA \right| \le$$

$$\mathrm{Const} \cdot \left(r \cdot \mathrm{Size}(Q) \max_{z \in R} |D^2 g(z)| + r^3 \max_{z \in R} |D^3 g(z)| \right).$$

The *energy* of a function $u\colon \widetilde{\mathcal{R}} \to \mathbb{R}$ is $E_{\mathcal{R}}(u) := \int_{\mathcal{R}} |\nabla u|^2 dA$.
The *energy* of a function $u\colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}$ is

$$E_{\mathcal{T}}(u) := \sum_{e\in\mathcal{T}^1} \frac{\cot\alpha_e + \cot\beta_e}{2}(u(h_e) - u(t_e))^2 = E_{\mathcal{R}}(I_{\mathcal{T}}u),$$

where $I_{\mathcal{T}}u$ is the piecewise-linear interpolation of u.

Energy Convergence Lemma for Abelian Integrals.
$\forall\delta > 0$ and $\forall u\colon \widetilde{\mathcal{R}} \to \mathbb{R}$ — smooth multi-valued function $\exists \mathrm{Const}_{u,\delta,\mathcal{R}}, \mathrm{const}_{u,\delta,\mathcal{R}} > 0$ such that for any triangulation $\mathcal{T}$ of $\mathcal{R}$ with the maximal edge length $h < \mathrm{const}_{u,\delta,\mathcal{R}}$ and with the minimal face angle $> \delta$ we have

$$|E_{\mathcal{T}}(u\,|_{\widetilde{\mathcal{T}}^0}) - E_{\mathcal{R}}(u)| \le \mathrm{Const}_{u,\delta,\mathcal{R}} \cdot \begin{cases} h, & \text{if } \gamma_{\mathcal{R}} > 1/2; \\ h|\log h|, & \text{if } \gamma_{\mathcal{R}} = 1/2; \\ h^{2\gamma_{\mathcal{R}}}, & \text{if } \gamma_{\mathcal{R}} < 1/2. \end{cases}$$

Energy Conservation Principle. *Let f be a discrete Abelian integral of the 1st kind with periods $A_1, \dots, A_g, B_1, \dots, B_g$. Then $E_\mathcal{T}(\mathrm{Re} f) = -\mathrm{Im} \sum_{k=1}^{g} A_k \bar{B}_k$.*

Corollary. $\exists$ *discrete harmonic $u_{\mathcal{T},A_1,\dots,A_g,B_1,\dots,B_g} : \widetilde{\mathcal{T}}^0 \to \mathbb{R}$ with arbitrary periods $A_1, \dots, A_g, B_1, \dots, B_g \in \mathbb{R}$.*

Variational Principle. *$u_{\mathcal{T},A_1,\dots,A_g,B_1,\dots,B_g}$ has minimal energy among all the multi-valued functions with the same periods.*

Lemma. *$E_\mathcal{T}(u_{\mathcal{T},P})$ and $E_\mathcal{R}(u_{\mathcal{R},P})$ are quadratic forms in $P \in \mathbb{R}^{2g}$ with the block matrices*

$$E_\mathcal{T} := \begin{pmatrix} \mathrm{Re}\Pi_{\mathcal{T}^*}(\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1}\mathrm{Re}\Pi_\mathcal{T} + \mathrm{Im}\Pi_\mathcal{T} & (\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1}\mathrm{Re}\Pi_\mathcal{T} \\ \mathrm{Re}\Pi_{\mathcal{T}^*}(\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1} & (\mathrm{Im}\Pi_{\mathcal{T}^*})^{-1} \end{pmatrix},$$

$$E_\mathcal{R} := \begin{pmatrix} \mathrm{Re}\Pi_\mathcal{R}(\mathrm{Im}\Pi_\mathcal{R})^{-1}\mathrm{Re}\Pi_\mathcal{R} + \mathrm{Im}\Pi_\mathcal{R} & (\mathrm{Im}\Pi_\mathcal{R})^{-1}\mathrm{Re}\Pi_\mathcal{R} \\ \mathrm{Re}\Pi_\mathcal{R}(\mathrm{Im}\Pi_\mathcal{R})^{-1} & (\mathrm{Im}\Pi_\mathcal{R})^{-1} \end{pmatrix}.$$

Proof of the convergence of period matrices

Convergence Theorem for Period Matrices. $\forall\delta > 0$ $\exists \mathrm{Const}_{\delta,\mathcal{R}}, \mathrm{const}_{\delta,\mathcal{R}} > 0$ *such that for any triangulation* $\mathcal{T}$ *of* $\mathcal{R}$ *with the maximal edge length* $h < \mathrm{const}_{\delta,\mathcal{R}}$ *and with the minimal face angle* $> \delta$ *we have*

$$\|\Pi_{\mathcal{T}} - \Pi_{\mathcal{R}}\| \le \lambda(h) := \mathrm{Const}_{\delta,\mathcal{R}} \cdot \begin{cases} h, & \text{if } \gamma_{\mathcal{R}} > 1/2; \\ h|\log h|, & \text{if } \gamma_{\mathcal{R}} = 1/2; \\ h^{2\gamma_{\mathcal{R}}}, & \text{if } \gamma_{\mathcal{R}} < 1/2. \end{cases}$$

Proof modulo the above lemmas.

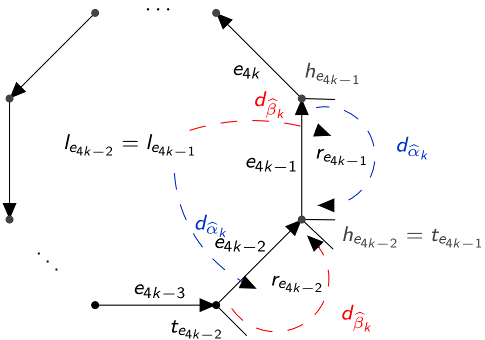
$$0 \le E_{\mathcal{T}}(u_{\mathcal{T},P}) - E_{\mathcal{R}}(u_{\mathcal{R},P}) \le E_{\mathcal{T}}(u_{\mathcal{R},P}\big|_{\widetilde{\mathcal{T}}^0}) - E_{\mathcal{R}}(u_{\mathcal{R},P}) \le \lambda(h)$$

$$\implies \|E_{\mathcal{T}} - E_{\mathcal{R}}\| \le \lambda(h) \implies \|\Pi_{\mathcal{T}} - \Pi_{\mathcal{R}}\| \le \lambda(h).$$

Lemma. *Let $u\colon \widetilde{\mathcal{T}}^0 \to \mathbb{R}$ and $u'\colon \widetilde{\mathcal{T}}^2 \to \mathbb{R}$ be multi-valued functions with periods $A_1, \dots, A_g, B_1, \dots, B_g$ and $A'_1, \dots, A'_g, B'_1, \dots, B'_g$, respectively. Then*

$$\sum_{e\in\mathcal{T}^1}(u'(l_e)-u'(r_e))(u(h_e)-u(t_e)) = \sum_{k=1}^{g}(A_kB'_k - B_kA'_k).$$

Proof plan.
1. Check the identity for the *canonical cell-decomposition*.
2. Perform edge subdivisions.



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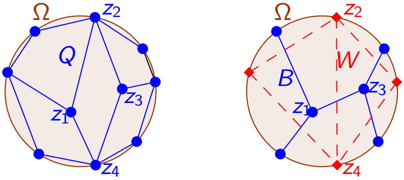
Open problems



Let Q be an *orthogonal lattice*. Set $c(z_1 z_3) := i\frac{z_2 - z_4}{z_1 - z_3} > 0$.
Consider a *random walk* on the graph B with transition probabilities proportional to $c(z_1 z_3)$.

Problem. The trajectories of a loop-erased random walk on B converge to SLE_2 curves in the scaling limit.

Remark. *Rhombic lattices*: Chelkak–Smirnov, 2008.

Open problems

Problem. Generalize Convergence Theorem to:

1. **nonorthogonal** quadrilateral lattices;
2. sequences of lattices with **unbounded** ratio of maximal and minimal edge lengths (to involve *adaptive meshes* for computer science applications);
3. **discontinuous** boundary values (for convergence of discrete harmonic measure, the Green function, the Cauchy and the Poisson kernels);
4. **mixed** boundary conditions;
5. **infinite** lattices and unbounded domains;
6. **higher** dimensions;
7. **other** elliptic PDE.

Acknowledgements

THANKS!