CHEKANOV-TYPE THEOREM FOR SPHERIZED COTANGENT BUNDLES

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Abstract. We generalise a Chekanov theorem to the space of cooriented contact elements.

Introduction

1. Chekanov-type theorem for sperization of a cotangent bundle

1.1. Contact structure on ST^*. We recall standard notions from contact geometry [AG]. Let B be a smooth manifold. Denote by 0_B the zero section of the cotangent bundle T^*B. The spherisation ST^*B of its cotangent bundle T^*B is a quotient space under the natural free action of a multiplicative group of positive real numbers $\mathbb{R}_+$ on $T^*B \setminus 0_B$, a positive number a transform a pair (q,p) $(q \in B, p \in T^*_qB)$ into the pair (q,ap). The space ST^*B is a smooth manifold of the dimension $2\dim B - 1$ and it carries a natural cooriented contact structure ξ defined as follows. Consider the Liouville 1-form $\lambda = pdq$ on T^*B, it defines a cooriented (by λ itself) hyperplane field $\lambda = 0$ on $T^*B \setminus 0_B$. That cooriented hyperplane field is invariant with respect to the action of $\mathbb{R}_+$ and it is tangent to orbits of the action. Hence the projection of that hyperplane field to ST^*B is a cooriented hyperplane field on ST^*B and it turns out to be a contact structure. We remark here, that there is no natural choice of a contact form on ST^*B.

1.2. Critical points and critical values of Legendrian manifolds in $J^1(B)$. We say that a point x of a Legendrian manifold is a critical point of Λ if it projects to a zero section 0_B under the natural projection $J^1(B) \to T^*B$. We say that a number a is a critical value of Λ if a equals to the value of u-coordinate of a critical poit of Λ .

1.3. Legendrian manifolds. The following notion generalise a notion of regular level set of a function on a manifold. Let $c \in \mathbb{R}$. Suppose that Λ is transverse to $T^*B \times c \subset T^*B \times \mathbb{R} = J^1(B)$. Note that it implies that c is not a critical value of Λ. Consider a manifold $L^c = \Lambda \cap T^*B \times c$. The intersection of L^c with $0_B \times c$ is empty and the restriction of natural

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projection $(T^*B \setminus 0_B) \times c \to ST^*B$ to L^c is a legendrian immersion. We denote the image of L^c by Λ^c and we say in that situation that Λ^c is a c-reduction of Λ.

1.4. Legendrian manifolds and generating families. Let us recall firstly the definition of generating family in the space of 1-jets of a function. Let B be a manifold, consider the space $J^1(B) = T^*B \times \mathbb{R}$ of one jets of functions on B. The space $J^1(B)$ is a contact manifold with the canonical contact structure given by the form $du - \lambda$, where λ is (a lift of) Liouville form on T^*B, u is the coordinate on the factor $\mathbb{R}$. A smooth bundle $E \to B$ and a generic function $F\colon E \to \mathbb{R}$ generates a (immersed) Legendrian submanifold $\Lambda_F \subset J^1(B)$ as follows. Consider a fiber of the bundle $E \to \mathbb{R}$, and a critical point of the restriction of the function F to this fiber. For a sufficiently generic function F the set C_F consisting of all such points is a smooth submanifold of the total space E (the genericity condition is that the equation $d_w F = 0$, where w is a local coordinate on a fiber of $E \to B$, satisfies the condition of the implicit function theorem). At any point z of C_F the differential $d_B F(z)$ of the function F along the base B is well defined. The rule $z \mapsto (z, d_B F(z), F(z))$ defines an immersion $l_F\colon C_F \to J^1(B)$ and its image is a Legendrian manifold Λ_F under definition. For a generating family F in a local trivialization $B \times W$ of E Λ_F is given by the formula:

$$\Lambda_F = \{(q,p,u) | \exists w_0 F_w(q, w_0) = 0, p = F_q(q, w_0), u = F(q, w_0)\},$$

where q, p are coordinates on T^*B.

Now we define a Legendrian (immersed) submanifold L_F in the space ST^*B starting from a smooth bundle $\pi\colon E \to B$ and a generic function $F\colon E \to \mathbb{R}$. For a point $b \in B$ we denote by $C_F^0(b)$ the zero level of the restriction of the function F to C_F. Genericity conditions are the following – C_F is a manifold in a neighborhood of C_F^0 and 0 is a regular value of C_F. Consider a map $C_F^0 \to ST^*B$, $z \mapsto (z, [d_B F(z)])$. That map is well defined since for $z \in C_F^0$ $d_B F(z) \neq 0$. Moreover, that map is a legendrian immersion and we denote its image by L_F. We will be interested in embedded Legendrian manifolds only.

We remark here that if F is a generating function for a manifold $\Lambda_F \subset J^1(B)$ then F is a generating function for a manifold $L_F \subset ST^*B$ if and only if Λ_F is transversal to the hypersurface $\{u = 0\}$ in $J^1(B)$. The last condition is equivalent to the condition of emptiness ???

Stabilization

Example

1.5. Legendrian isotopy lifting.

Lemma 1.1. *Let B be a closed manifold and $c \in \mathbb{R}$. Consider a compact Legendrian manifold $\Lambda \subset J^1(B)$ such that its c-reduction Λ^c is well defined and Λ^c is an embedded manifold. Let $L_{t, t\in[0,1]}$ be a legendrian isotopy of $\Lambda^c = L_0$. Than there exists a legendrian isotopy*

$\Lambda_{t,t\in[0,1]}, \Lambda_0 = \Lambda$ *such that for any* $t \in [0,1]$ *its* c*-reduction is defined and* $\Lambda_t^c = L_t$.

Proof. It is sufficient to prove the statement of lemma for $c = 0$. Consider the legendrian isotopy L_t. By isotopy extension theorem there exists a contact flow $\varphi_{t\in[0,1]}$, such that for any $t \in [0,1]$ $\varphi_t(L_0) = L_t$. Any contact isotopy of ST^*B lifts to a (homogeneous) Hamiltonian flow on $T^*B \setminus 0_B$. More precisely – consider a Hamiltonian $H_t\colon T^*B \setminus 0_B \to \mathbb{R}$ such that $H_t(ap, q) = aH_t(p, q)$ for any positive number a (we will say that such a Hamiltonian is homogeneous). Then the flow of such a Hamiltonian function is well defined for all values of t and projects to a contact flow on ST^*B. Moreover, any contact flow on ST^*B could be given as a projection of a unique Hamiltonian flow above.

We take a homogeneous Hamiltonian H_t corresponding to the flow φ_t. Consider a function $K_t(p, q, u) = H_t(p, q)$ on $(T^*B \setminus 0_B) \times \mathbb{R} \subset J^1(B)$ as a contact Hamiltonian (see [AG]) with respect to the contact form $du - \lambda$. Any set $(T^*B \setminus 0_B) \times c$ is invariant under the flow of generated by K_t and coincides on it with the flow of H_t under the forgetful identifications $T^*B \times c = T^*B$ that set with the flow of H_t. It follows from the explicit formula for the corresponding vector field: $\dot{u} = K - pK_p, \dot{p} = K_q - pK_u, \dot{q} = -K_p$ (see [AG]). u-component of that contact vector field equals to zero since K is homogeneous. Hence the flow ψ_t generated by K satisfy $\psi_t(\Lambda \cap (T^*B \setminus 0_B) = L_t$. But in general it is impossible to extend ψ_t to a flow on the whole space $J^1(B)$ so we will change the function K_t. Let us fix an arbitrary smooth function $\widetilde{H}_t\colon T^*B \to \mathbb{R}$ coinciding with H_t in a neighbourhood of infinity. Denote by P_t the function $P_t(p, q, u) = \widetilde{H}_t(p, q)$ and by P_t^C $(C \in \mathbb{R}_+)$ the function $P_t^C(p, q, u) = \frac{1}{C}P_t(Cp, q, u)$. We claim that for sufficiently big C the legendrian isotopy of Λ generated by the contact flow Ψ_t^C of P_t^C satisfyes to the claim of lemma.

Let us fix a number a such that absolute value of any critical value of Λ is bigger then $2a$. Denote by $X \subset \Lambda$ the subset formed by all points such that an absolute value of u-coordinate is at most a, by Y we denote the closure of its complement $\Lambda \setminus X$. The set X is a compact set and contained in $(T^*B \setminus 0_B) \times \mathbb{R}$. Take a neighborhood $U \subset T^*B$ of zero section, the support of $P_t^C - K_t$ containes in $U \times \mathbb{R}$ for sufficiently big C. Hence, for sufficiently big C $\Psi_t^C(X) = \psi_t(X)$ for all $t \in [0,1]$. It remains to show that for sufficiently big C u-coordinate of any point in $\Psi_t^C(Y)$ could not be zero and hence zero reduction of $\Psi_t^C(\Lambda)$ is L_t. The coordinate u changes under an action of a contact Hamiltonian P according to the low: $\dot{u} = P_t^C - p\frac{\partial P_t^C}{\partial p}$. So it is sufficient to show that the speed uniformly tends to zero. The following general consideration finishes the proof.

Consider a smooth vector bundle V over closed manifold M. We denote by $M(c)$ fiberwise multiplication by c. We say that a smooth function on V is positively homogeneous of degree 1 at infinity if it coincide with a continuous positively homogeneous (i.e. $1/c(M(c))^*$-invariant for any positive c) of degree 1 function up to a sum with a compactly supported continuous function. Let v be a vertical vector field coinciding with Euler vector field on each fiber. Consider an operator D sending a function g on V to $g - L_v g$. For a positively homogeneous function f the function Df is a compactly supported function. Denote by f^c the function $1/c(M(c))^* f$, i.e. for any $x \in V$ $f^c(x) = 1/cf(M(c)x)$.

Lemma 1.2. *For any smooth positively homogeneous of degree* 1 *at infinity function f, C^0-norm of $D(1/c(M(c))^* f)$ tends to zero while $C \to +\infty$.*

Proof. Indeed, $D(\frac{1}{c}(M(c))^* f) = \frac{1}{c}(M(c))^* D(f)$. Hence C^0-norm of $D(1/c(M(c))^* f)$ equals to C^0-norm of f divided by C. □

1.6. Chekanov-type theorem. Consider the space ST^*B of cooriented contact elements on a closed manifold B.

Theorem 1.3. *Let $\{L_t\}_{t \in [0,1]}$ be a legendrian isotopy of a compact Legendrian manifold $L_0 \subset ST^*B$. Suppose L_0 is given by a generating family $F\colon E \to \mathbb{R}$, for a smooth compact fibration $E \to B$. Then there exists $N \in \mathbb{Z}_+$, such that L_t is given by a generating family $G_t\colon E \times \mathbb{R}^N \to \mathbb{R}$ of the form:*

$$G_t(e,q) = F(e) + Q(q) + f_t(e,q)$$

for a nondegenerate quadratic form Q on $\mathbb{R}^N$ and compactly supported function f_t such that $f_0 = 0$.

The (generalized) Chekanov theorem [Ch, P] has almost the same statement – ST^*B is replaced by $J^1(B)$.
Proof. Proof is a reduction to (generalised) Chekanov theorem. By legendrian isotopy lifting lemma we reduce the problem to $J^1(B)$-case. □

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